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On the nodal line of the second eigenfunction of elliptic operators in two dimensions

By *Thomas Kappeler* at Philadelphia and *Bernhard Ruf* at Milano

1. Introduction

It is a well-known fact that the n-th eigenfunction of the Sturm-Liouville eigenvalue equation $-u''(x)+q(x)\,u(x)=\lambda u(x)$, $x \in (0, 1)$, $u(0)=u(1)=0$, has exactly $n-1$ nodes (i.e. non-degenerate zeroes), see e.g. [CH].

For the corresponding equation in higher dimensions it is much more complicated to obtain general statements on the zero sets of the eigenfunctions. Let us concentrate on the equation in two dimensions: Let Ω be a bounded and smooth domain in $\mathbb{R}^2$ and q a potential in $C^\infty(\Omega)$, and consider

$$(1) \qquad \begin{aligned} -\Delta u + qu &= \lambda u, \quad \text{in} \quad \Omega \\ u &= 0, \quad \text{on} \quad \partial\Omega. \end{aligned}$$

In analogy to the one-dimensional case let $k(n)$ denote the number of connected components of $\Omega \backslash Z_n$, where Z_n is the nodal set of the n-th eigenfunction u_n of (1), i.e.

$$Z_n = \{x \in \Omega;\, u_n(x) = 0\}.$$

We recall some of the known results concerning equation (1). In [C] one finds the Courant nodal domain theorem, which states that $k(n) \leqq n$ (cf. also [CH]). Pleijel [P] has proved in addition that

$$(2) \qquad \limsup_{n \to \infty} \frac{k(n)}{n} \leqq \frac{4}{j^2} < 0.7,$$

where j denotes the smallest zero of the 0-th Bessel function. From (2) one concludes that $k(n)=n$ can occur only a finite number of times. There are even examples [S] where the case $k(n)=2$ arises infinitely many times.

In recent years there has been a renewed interest in the properties of nodal sets, see e.g. [BG], [B], [Be], [DF], [HS], [Y].

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In [BG] Brüning and Gromes treat the equation

$$(3)\qquad \begin{aligned} -\Delta u &= \lambda u \quad \text{in} \quad \Omega \\ u &= 0 \quad \text{on} \quad \partial\Omega, \end{aligned}$$

where Ω is a bounded domain of genus $s+1$ with area F and circumference U. They prove the following lower bound for the length $l(n)$ of the nodal set of the n-th eigenfunction

$$l(n)+\frac{U}{2} \geqq \frac{F\sqrt{\lambda_n}}{2j} - \pi\frac{j}{2\sqrt{\lambda_n}}(s-1),$$

where λ_n denotes the n-th eigenvalue and j is the smallest zero of the first Bessel function. With Weyl's asymptotic formula $\lambda_n \sim \frac{4\pi n}{F}$ they then get the following asymptotic lower bound for the length of the nodal line of the n-th eigenfunction

$$\liminf_{n\to\infty} \frac{l(n)+\frac{U}{2}}{F\cdot n} \geqq \frac{\pi}{j^2}.$$

Brüning [B] has generalized this result to compact Riemannian manifolds of dimension 2.

In this paper we show that for equation (1) there exists no upper bound for the length of the nodal line of the *second* eigenfunction independently of the potential. We prove

Theorem 1. *Let Ω satisfy an interior sphere condition at each point $x \in \partial\Omega$, and let Γ be a Lipschitz continuous curve in $\bar{\Omega}$ which divides Ω into exactly two components. Then there exists for every given $\varepsilon > 0$ a potential q_ε in $L^\infty(\Omega)$ such that*

$$Z_2(q_\varepsilon) \subset [\Gamma]_\varepsilon$$

where $Z_2(q_\varepsilon)$ denotes the nodal line of the second *eigenfunction of* (1) *with $q = q_\varepsilon$, and $[\Gamma]_\varepsilon$ is defined by $[\Gamma]_\varepsilon = \{x \in \Omega;\ \operatorname{dist}(x, \Gamma) \leqq \varepsilon\}$.*

Note that for any given number $L > 0$ one can choose a curve Γ of length larger than $L+1$ satisfying the conditions of Theorem 1. By Theorem 1 it follows that for $\varepsilon > 0$ sufficiently small the length of $Z_2(q_\varepsilon)$ is larger than L. Hence we have the following

Corollary. *There exists no upper bound to the length of the nodal line of the* second *eigenfunction uniformly for all potentials.*

In section 3 we extend the theorem in two directions: First, we give a corresponding result concerning the first N eigenfunctions, and second, we generalize the statements to general second order, symmetric and uniformly elliptic differential operators.

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We mentioned in the beginning that in one dimension the n-th eigenfunction u_n can be characterized by its $n-1$ nodal points. On the other hand, the n-th eigenvalue has also a variational characterization, namely

$$\lambda_n = \min_{F_n} \max_{0 \neq v \in E_n} \frac{\int |v'|^2 \, dx}{\int |v|^2 \, dx} \tag{4}$$

where $F_n = \{E_n = n\text{-dimensional subspace of } H_0^1((0, 1))\}$. Hence, there is a correspondence between the topological characterization (4) of λ_n and the geometric characterization of u_n by the number of nodal points. Many important consequences arise from this relation.

For the problem in higher dimensions one has also a variational characterization of the eigenvalues, but no geometric characterization of the eigenfunctions is known. Our result indicates that such a characterization (if it exists) cannot depend solely on the length of the nodal set.

To prove Theorem 1 we construct explicitely the potential q_ε. The idea of the construction is to study carefully an approximation of the variational problem for the Dirichlet integral on the space $H_\Gamma = \{u \in H_0^1(\Omega);\, u_{|\Gamma} = 0\}$. This approximation relies on the heuristic observation that adding a potential which is large near Γ forces the minimizer to be small near Γ.

2. Proof of Theorem 1

We first introduce some notation. Let us denote by Ω a bounded domain in $\mathbb{R}^2$ with a C^1 boundary $\partial\Omega$ and with an interior sphere condition at each point x of $\partial\Omega$. Let $\Gamma: [0, 1] \to \bar{\Omega}$ be a Lipschitz continuous (and possibly closed) curve such that $\Omega \backslash \Gamma = \Omega_1 \cup \Omega_2$ consists of exactly two components. Furthermore, let

$$\Omega_{3n} = \left\{ x \in \Omega_1;\ \operatorname{dist}(x, \Gamma) < \frac{1}{n} \right\}$$

and set $\Omega_{1n} = \Omega_1 \backslash \bar{\Omega}_{3n}$. With $\mu_{n1} < \mu_{n2} \leqq \cdots$ we denote the Dirichlet eigenvalues corresponding to Ω_{1n}, i.e. the eigenvalues of $-\Delta y = \lambda y$ on Ω_{1n} with $y|_{\partial\Omega_{1n}} = 0$. Let $v_{n1}, v_{n2}, \ldots$, with $v_{n1} > 0$ on Ω_{1n}, denote a corresponding orthonormal set of eigenfunctions. Moreover, let $\mu_1 < \mu_2 \leqq \cdots$ and $\nu_1 < \nu_2 \leqq \nu_3 \leqq \cdots$ denote the Dirichlet eigenvalues of Ω_1 and Ω_2, respectively, with corresponding sets of eigenfunctions $v_1, v_2, \ldots$, and $w_1, w_2, \ldots$. By the monotone dependence of the eigenvalues on the domain (see Courant-Hilbert [CH]) the sequences $(\mu_{nk})_{n \in \mathbb{N}}$, $k \in \mathbb{N}$, are monotone increasing and

$$\lim_{n \to \infty} \mu_{nk} = \mu_k, \; k \in \mathbb{N}.$$

Let us assume without loss of generality that $\mu_1 < \nu_1$ (if $\mu_1 > \nu_1$ interchange Ω_1 and Ω_2, if $\mu_1 = \nu_1$, perturb slightly Γ such that Ω_1 becomes bigger and Ω_2 smaller). Now we choose a positive convergent sequence $(a_n)_{n \in \mathbb{N}}$ such that for $\eta_{nk} = \mu_{nk} + a_n$ and

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$\eta_k = \mu_k + a$, where $a = \lim_{n\to\infty} a_n$, the following inequalities hold:

$$\eta_{n1} < \nu_1 < \eta_{n2}, \quad n \in \mathbb{N}$$

and

$$\eta_1 = \nu_1 < \eta_2.$$

We now introduce the following sequence $(q_n)_{n\in\mathbb{N}}$ of potentials, i.e. functions defined on Ω, by

$$q_n(x) := r_n 1_{\Omega_{3n}}(x) + a_n 1_{\Omega_{1n}}(x),$$

where $r_n > 0$ are real numbers which will be specified later.

We consider the eigenvalue equation

$$(5) \qquad -\Delta y + q_n y = \lambda y, \ \ y \in H_0^1(\Omega) \cap H^2(\Omega).$$

Let $\lambda_{n1} < \lambda_{n2} \leqq \lambda_{n3} \leqq \cdots$ denote the eigenvalues of (5), and $u_{n1}, u_{n2}, \ldots$ a corresponding set of $L^2(\Omega)$-orthonormal eigenfunctions. With the notations introduced above we get

Lemma 1. *The following estimates hold independently of $n \in \mathbb{N}$:*

1) $0 \leqq \lambda_{n2} \leqq \nu_1$,

2) $\|\nabla u_{n2}\|^2_{L^2(\Omega)} \leqq \nu_1$,

3) $\|\Delta u_{n2}\|_{L^2(\Omega_2)} \leqq \nu_1$,

4) $\|\Delta u_{n2}\|_{L^2(\Omega_{1n})} \leqq \sup\{\nu_1, a_n;\ n \in \mathbb{N}\}$.

Remark. All the bounds are independent of the sequence $(r_n)_{n\in\mathbb{N}}$.

Proof. By the variational characterization of the second eigenvalue we get

$$\lambda_{n2} = \inf\{\|\nabla u\|^2_{L^2(\Omega)} + r_n \|u\|^2_{L^2(\Omega_{3n})} + a_n \|u\|^2_{L^2(\Omega_{1n})};$$
$$u \in H_0^1(\Omega),\ \|u\|_{L^2(\Omega)} = 1,\ \langle u, u_{n1}\rangle = 0\},$$

where $\langle\cdot,\cdot\rangle$ denotes the usual inner product in $L^2(\Omega)$. Let us denote by $\bar v_{n1}$ and $\bar w_1$ the functions v_{n1} and w_1, respectively, extended by zero to the whole of Ω. Choose δ_{n1} and δ_{n2} with $\delta_{n1}^2 + \delta_{n2}^2 = 1$ and $\langle \delta_{n1}\bar v_{n1} + \delta_{n2}\bar w_1, u_{n1}\rangle = 0$. As $y_n := \delta_{n1} v_{n1} + \delta_{n2}\bar w_1$ is in $H_0^1(\Omega)$ and $\|y_n\|_{L^2(\Omega)} = 1$ one gets from the variational characterization of λ_{n2}

$$\lambda_{n2} \leqq \delta_{n1}^2(\|\nabla v_{n1}\|^2_{L^2(\Omega_{1n})} + a_n\|v_{n1}\|^2_{L^2(\Omega_{1n})}) + \delta_{n2}^2\|\nabla w_1\|^2_{L^2(\Omega_2)}$$
$$\leqq \delta_{n1}^2 \eta_{n1} + \delta_{n2}^2 \nu_1 \leqq \nu_1.$$

So 1) and 2) follow. 4) follows from the fact that u_{n2} satisfies $-\Delta u_{n2} = (\lambda_{n2} - a_n) u_{n2}$ on Ω_{1n}. On Ω_2 u_{n2} satisfies $-\Delta u_{n2} = \lambda_{n2} u_{n2}$ and thus 3) follows.

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Using lemma 1 we may and do assume for the rest of this paper that the sequence $(q_n)_{n\in\mathbb{N}}$ has been chosen in such a way that

(a) $(\lambda_{n2})_{n\in\mathbb{N}}$ converges; the limit is denoted by λ_2;

(b) $(u_{n2})_{n\in\mathbb{N}}$ converges weakly in $H_0^1(\Omega)$; the limit function in $H_0^1(\Omega)$ is denoted by u_2;

c) $\lim_{n\to\infty} u_{n2}=u_2$ weakly in $H^2(\Omega_2)$ and weakly in $H^2(\Omega_{1m})$ for $m=1, 2, \ldots$.

In the following lemma we show that the limit function u_2 can be identified if we choose suitably the sequence $(r_n)_{n\in\mathbb{N}}$.

Denote by $\bar{v}_1$ and $\bar{w}_1$ the functions v_1 and w_1, respectively, extended by zero to the whole of Ω. Let d_n be the interpolation constant (see [LM]) of the following type

$$(6)\qquad \|u\|_{H^{1/2}(\Omega_{3n})} \leqq d_n \|u\|_{L^2(\Omega_{3n})}^{1/2} \cdot \|u\|_{H^1(\Omega_{3n})}^{1/2} \; (u \in H^1(\Omega_{3n})).$$

Moreover, let c_n be the embedding constant for

$$H^{1/2}(\Omega_{3n}) \hookrightarrow L^2(\partial\Omega_{3n}).$$

Lemma 2. *If $(r_n)_{n\in\mathbb{N}}$ satisfies $\lim_{n\to\infty} \frac{c_n d_n}{r_n^{1/4}}=0$, then there exist constants α and β with $\alpha^2+\beta^2=1$ such that*

1) $\lambda_2=\nu_1(=\eta_1)$,

2) $u_2=\alpha\bar{v}_1+\beta\bar{w}_1$,

3) $\lim_{n\to\infty} u_{n2}=\alpha v_1$ *strongly in* $H^k(\Omega_{1m})$ *(all m and k),*

4) $\lim_{n\to\infty} u_{n2}=\beta w_1$ *strongly in* $H^k(\Omega_2)$ *(all k).*

Proof. As $H_0^1(\Omega)$ is compactly imbedded in $L^2(\Omega)$ one concludes that $\lim_{n\to\infty} u_{n2}=u_2$ strongly in $L^2(\Omega)$. On Ω_2 we have $-\Delta u_{n2}=\lambda_{n2}u_{n2}$ and thus $-\Delta u_2=\lambda_2 u_2$ and the convergence holds strongly in $H^2(\Omega_2)$. Similarly one argues for Ω_{1m}. So 1), 3) and 4) follow immediately from 2).

To prove 2) we first show that $u_2|_{\Omega_2}\in H_0^1(\Omega_2)$. Since $\lim_{n\to\infty} u_{n2}=u_2$ weakly in $H^1(\Omega_2)$ we conclude by the trace embedding theorem (see [LM]) that $u_2=\lim_{n\to\infty} u_{n2}$ weakly in $H^{1/2}(\partial\Omega_2)$; but $H^{1/2}(\partial\Omega_2)\hookrightarrow L^2(\partial\Omega_2)$ compactly, and thus $u_2=\lim_{n\to\infty} u_{n2}$ strongly in $L^2(\partial\Omega_2)$. Since $\Omega\cap\partial\Omega_2\subset\partial\Omega_{3n}$ we get the following estimate:

$$\begin{aligned}\|u_{n2}|_{\Omega\cap\partial\Omega_2}\|_{L^2(\Omega\cap\partial\Omega_2)} &\leqq \|u_{n2}|_{\partial\Omega_{3n}}\|_{L^2(\partial\Omega_{3n})}\\ &\leqq c_n\|u_{n2}\|_{H^{1/2}(\Omega_{3n})}\\ &\leqq c_n d_n \|u_{n2}\|_{L^2(\Omega_{3n})}^{1/2}\|u_{n2}\|_{H^1(\Omega_{3n})}^{1/2};\end{aligned}$$

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where c_n and d_n are the imbedding and interpolation constants defined above. The two norms in the last term can be estimated as follows: By lemma 1 we know that

$$\|u_{n2}\|^2_{H^1(\Omega_{3n})} \leqq \|u_{n2}\|^2_{H^1_0(\Omega)} \leqq \nu_1 + 1 .$$

Using

$$\int_\Omega |\nabla u_{n2}|^2 dx + a_n \int_{\Omega_{1n}} u_{n2}^2 dx + r_n \int_{\Omega_{3n}} u_{n2}^2 dx = \lambda_{n2} \int_\Omega u_{n2}^2 dx \leqq \nu_1$$

one concludes that

$$\|u_{n2}\|^2_{L^2(\Omega_{3n})} = \int_{\Omega_{3n}} u_{n2}^2 dx \leqq \frac{\nu_1}{r_n} .$$

Thus we obtain

$$\|u_{n2}|_{\Omega \cap \partial\Omega_2}\|_{L^2(\Omega \cap \partial\Omega_2)} \leqq c_n d_n \left(\frac{\nu_1}{r_n}\right)^{1/4} (1+\nu_1)^{1/4} .$$

By the choice of the sequence $(r_n)_{n \in \mathbb{N}}$ we get

$$u_2|_{\Omega \cap \partial\Omega_2} = \lim_{n \to \infty} u_{n2}|_{\Omega \cap \partial\Omega_2} = 0 \quad \text{strongly in} \quad L^2(\Omega \cap \partial\Omega_2).$$

But u_2 is in $H^1_0(\Omega)$ and thus $u_{2|\partial\Omega_2} = 0$ in $L^2(\partial\Omega_2)$.

Since $\Omega \cap \partial\Omega_2 = \Omega \cap \partial\Omega_1$ it follows that u_2 is also in $H^1_0(\Omega_1)$. On $\Omega_{1m} (m \in \mathbb{N})$ the equation $-\Delta u_2 = (\lambda_2 - a) u_2$ holds. As u_2 is in $H^1_0(\Omega)$ we conclude that $-\Delta u_2 = (\lambda_2 - a) u_2$ holds on $\bigcup_{n \in \mathbb{N}} \Omega_{1n} = \Omega_1$. To summarize: $u_2 \in H^1_0(\Omega)$ satisfies

$$\begin{aligned} -\Delta u_2 + a u_2 = \lambda_2 u_2 \quad &\text{on} \quad \Omega_1, \quad u_2|_{\Omega_1} \in H^1_0(\Omega_1), \\ -\Delta u_2 = \lambda_2 u_2 \quad &\text{on} \quad \Omega_2, \quad u_2|_{\Omega_2} \in H^1_0(\Omega_2). \end{aligned}$$

From $\|u_{n2}\|_{L^2(\Omega)} = 1$, $n \in \mathbb{N}$, we conclude that $\|u_2\|_{L^2(\Omega)} = 1$ and hence $u_2 \neq 0$. Then $u_2|_{\Omega_1} \neq 0$ or $u_2|_{\Omega_2} \neq 0$. Let us assume that $u_2|_{\Omega_2} \neq 0$. (The case $u_2|_{\Omega_1} \neq 0$ is treated similarly.) Due to the fact that $\lambda_2 \leqq \nu_1$ and ν_1 is the lowest Dirichlet eigenvalue on Ω_2 we conclude that $\lambda_2 = \nu_1$. Together with $\eta_1 = \nu_1$ this implies that $u_2 = \alpha \bar{v}_1 + \beta \bar{w}_1$ with $\alpha^2 + \beta^2 = 1$.

Let us now look at the nodal lines of u_{n2}. We assume that the constants α and β in $u_2 = \alpha \bar{v}_1 + \beta \bar{w}_1$ satisfy $\alpha \geqq 0$ and $\beta \leqq 0$. The case $\alpha \leqq 0$ and $\beta \geqq 0$ is treated similarly. We first consider the case where $\alpha > 0$ and $\beta < 0$. With this assumption we prove in the following lemma that the nodal line Z_n of u_{n2} approaches Γ (except possibly near the boundary $\partial\Omega$).

Lemma 3. *Assume that $u_2 = \alpha \bar{v}_1 + \beta \bar{w}_1$ with $\alpha > 0$ and $\beta < 0$. Let U, V be arbitrary open sets with $\bar{U} \subset \Omega_1$ and $\bar{V} \subset \Omega_2$. Then there exists a number n_0 such that $\bar{U} \cap Z_n = \emptyset$ and $\bar{V} \cap Z_n = \emptyset$, for all $n \geqq n_0$.*

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Proof. From lemma 2 we know that $\lim_{n\to\infty} u_{n2} = u_2$ strongly in $H^2(U)$. But $H^2(U) \hookrightarrow C(\bar{U})$ is a compact embedding and thus $\lim_{n\to\infty} u_{n2} = u_2$ in $C(\bar{U})$. By the strict positivity of the first eigenfunctions v_1, w_1 on Ω_1, Ω_2, respectively, we get

$$\min\{u_2(x);\, x\in\bar{U}\} = \min\{\alpha v_1(x);\, x\in\bar{U}\} \geqq \varepsilon > 0,$$

and hence there exists an $n_0(\alpha)$ such that

$$u_{n2}(x) \geqq \frac{\varepsilon}{2}, \quad \text{for all} \quad n \geqq n_0(\alpha),\ x\in\bar{U}.$$

This means that $\bar{U}\cap Z_n = \emptyset$ for all $n\geqq n_0(\alpha)$. Similarly one can argue for V.

If Γ is a closed curve lying in Ω, then we can easily complete the proof of the theorem. In fact, in this case the whole nodal line Z_n of u_{n2} lies either near the boundary $\partial\Omega$ or near Γ (for n sufficiently large). The first possibility cannot occur. To see it, denote by D_n the domain lying between Z_n and $\partial\Omega$. Then $-\Delta u_{n2} = \lambda_{n2} u_{n2}$ in D_n with $u_{n2} = 0$ on ∂D_n (if $D_n \subset \Omega_1$ a term $a_n u_{n2}$ has to be added on the left). Assuming that Z_n lies near the boundary $\partial\Omega$ the area $|D_n|$ gets arbitrarily small, contradicting that $\lambda_{n2} \leqq v_1$ for all $n\in\mathbb{N}$. We remark that in this case no interior sphere condition on $\partial\Omega$ is needed.

If the curve Γ meets the boundary $\partial\Omega$ then we need an additional argument to complete the proof of the theorem (still in the case $\alpha>0$ and $\beta<0$).

Lemma 4. *For every $\varepsilon>0$ there exists a number n_0 and constants $\delta, \gamma>0$ such that*

$$|u_{n2}(x)| \geqq \gamma\cdot \operatorname{dist}(x, \partial\Omega\backslash[\Gamma]_\varepsilon)\ (x\in([\partial\Omega]_\delta\cap\Omega)\backslash[\Gamma]_\varepsilon).$$

Proof. By lemma 2 and by using Sobolev's embedding theorem

$$H^3(\Omega_2\cup\Omega_{1m}) \hookrightarrow C^1(\overline{\Omega_2\cup\Omega_{1m}}),$$

it follows that

$$\lim_{n\to\infty}\ \sup_{x\in\partial\Omega\backslash[\Gamma]_\varepsilon} |\partial_\nu(u_{n2}(x) - u_2(x))| = 0,$$

where ν denotes the inside normal to $\partial\Omega$, and ∂_ν the derivative in the direction ν. From [GT], p. 33, we conclude, recalling that $\alpha v_1>0$ in Ω_1 and $\beta w_1<0$ in Ω_2, and using the interior sphere condition on $\partial\Omega$, that for a given $\varepsilon>0$ there exists a $\gamma>0$ such that

$$\alpha\partial_\nu v_1(x) \geqq 2\gamma, \qquad \text{for all} \quad x\in\partial\Omega_1\backslash[\Gamma]_\varepsilon,$$
$$\beta\partial_\nu w_1(x) \leqq -2\gamma, \quad \text{for all} \quad x\in\partial\Omega_2\backslash[\Gamma]_\varepsilon.$$

Hence there exists a number n_0 and a constant $\delta>0$ such that

$$|u_{n2}(x)| \geqq \gamma\cdot\operatorname{dist}(x, \partial\Omega\backslash[\Gamma]_\varepsilon)$$

for $n\geqq n_0$ and $x\in([\partial\Omega]_\delta\cap\Omega)\backslash[\Gamma]_\varepsilon$.

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In the case that $u_2 = \alpha \bar{v}_1 + \beta \bar{w}_1$ with $\alpha > 0$ and $\beta < 0$ this completes the proof of the theorem, since by lemma 3 and 4 there exists a number k_0 such that $u_{k2} \neq 0$ for all $x \in \Omega \backslash [\Gamma]_\varepsilon$, for all $k \geqq k_0$.

In the case that $\alpha = 0$ or $\beta = 0$ we need some additional results to investigate the nodal lines. We consider here the case that $\alpha = 0$. The case $\beta = 0$ is treated similarly.

Let $u_{n2}^+ := \max\{u_{n2}, 0\}$, and $f_n := u_{n2}^+ / \|u_{n2}^+\|_{L^2(\Omega)}$. Due to $\langle u_{n1}, u_{n2} \rangle = 0$ we have $\|u_{n2}^+\|_{L^2(\Omega)} \neq 0$ for all $n \in \mathbb{N}$ and thus f_n is well defined.

Lemma 5. $\lim\limits_{n \to \infty} f_n = \bar{v}_1$ *weakly in* $H_0^1(\Omega)$ *and a.e. in* Ω.

Proof. We have $f_n \in H_0^1(\Omega)$, for all $n \in \mathbb{N}$ (see [LM]). From

$$-\Delta u_{n2} + q_n u_{n2} = \lambda_{n2} u_{n2}$$

we get by scalar multiplication with f_n and division by $\|u_{n2}^+\|$

$$\int_\Omega |\nabla f_n|^2 \, dx + \int_\Omega q_n f_n^2 \, dx = \lambda_{n2} \int_\Omega f_n^2 \, dx \leqq \nu_1 \tag{7}$$

and hence $\|f_n\|_{H_0^1(\Omega)}^2 \leqq \nu_1 + 1$.

To prove the lemma it is sufficient to show that for every subsequence of $(f_n)_{n \in \mathbb{N}}$ there exists another one which converges weakly to $\bar{v}_1$ in $H_0^1(\Omega)$. As $(f_n)_{n \in \mathbb{N}}$ is bounded in $H_0^1(\Omega)$ there exists a subsequence of $(f_n)_{n \in \mathbb{N}}$, again denoted by $(f_n)_{n \in \mathbb{N}}$, and a function f in $H_0^1(\Omega)$ such that $f = \lim\limits_{n \to \infty} f_n$ weakly in $H_0^1(\Omega)$.

According to lemma 3 $f|_{\bar{V}} = 0$ for every open set V with $\bar{V} \subset \Omega_2$. Thus $f|_{\Omega_2} = 0$. Since $f|_{\Omega_2} \in H^1(\Omega_2)$ we conclude by the trace embedding theorem that $f|_{\partial \Omega_2} = 0$ in $L^2(\partial \Omega_2)$. Since $f \in H_0^1(\Omega)$ this implies that $f|_{\partial \Omega_1} = 0$ in $L^2(\partial \Omega_1)$, also. This shows that f is in $H_0^1(\Omega_1)$.

From (7) we get

$$\int_\Omega |\nabla f_n|^2 \, dx < \lambda_{n2} - a_n \int_{\Omega_{1n}} f_n^2 \, dx.$$

Now $\Omega_{1n} \subset \Omega_{1(n+1)} \subset \cdots$ with $\bigcup\limits_{n \in \mathbb{N}} \Omega_{1n} = \Omega_1$, and $f = \lim\limits_{n \to \infty} f_n$ strongly in $L^2(\Omega)$, thus $\lim\limits_{n \to \infty} \int_{\Omega_{1n}} f_n^2 \, dx = \int_{\Omega_1} f^2 \, dx = 1$, where the last equality follows from $\int_\Omega f^2 \, dx = 1$ and $f|_{\Omega_2} = 0$. Using $\lim\limits_{n \to \infty} \lambda_{n2} = \nu_1 = \eta_1$ and $\lim\limits_{n \to \infty} a_n = a$ we obtain

$$\int_{\Omega_1} |\nabla f|^2 \, dx \leqq \int_\Omega |\nabla f|^2 \, dx \leqq \limsup_{n \to \infty} \int_\Omega |\nabla f_n|^2 \, dx \leqq \eta_1 - a = \mu_1.$$

Since μ_1 is the lowest Dirichlet eigenvalue of Ω_1 and $f \in H_0^1(\Omega_1)$ with $\|f\|_{L^2(\Omega_1)} = 1$ and $f > 0$ we conclude that $\int_{\Omega_1} |\nabla f|^2 \, dx = \mu_1$ and $f = \bar{v}_1$.

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Let us introduce the sets $F_k = \{x \in \Omega;\ u_{k2}(x) > 0\}$ and

$$B_n := \bigcap_{k \geqq n} F_k = \{x \in \Omega;\ u_{k2}(x) > 0 \text{ for all } k \geqq n\}.$$

We get the following

Corollary 6. *$\Omega_1 \setminus \bigcup_{n \in \mathbb{N}} B_n$ is a null set.*

Proof. x is an element of $\Omega_1 \setminus \bigcup_{n \in \mathbb{N}} B_n$ iff x is in Ω_1 and $x \notin \Omega_1 \cap B_n$ for all $n \in \mathbb{N}$, that is for each $n \in \mathbb{N}$ there exists a $k(n) \geqq n$ such that $x \notin \Omega_1 \cap F_{k(n)}$, i.e. $u^+_{k(n)2} = 0$, $n \in \mathbb{N}$. But then $f_{k(n)}(x) = 0$ $(n \in \mathbb{N})$ and thus $\lim_{n \to \infty} f_{k(n)}(x) = 0$. As $\lim_{n \to \infty} f_n = \bar{v}_1$ a.e. and $v_1 > 0$ in Ω_1 the claim follows.

Let us now fix m and introduce the functions

$$g_{mn} = u_{n2}\, 1_{\Omega_{1m} \cup F_n}, \quad h_{mn} = g_{mn} / \| g_{mn} \|_{L^2(\Omega)}.$$

Observe that $\| g_{mn} \|_{L^2(\Omega)} \neq 0$ and hence h_{mn} is well-defined.

Lemma 7. *Fix $m \geqq 1$. Then*

1) $\lim_{n \to \infty} h^+_{mn} = c \bar{v}_1$ *weakly in $H^1_0(\Omega)$ and a.e. with $0 \leqq c \leqq 1$,*

2) $\lim_{n \to \infty} h_{mn} = \bar{v}_1$ *strongly in $L^2(\Omega)$,*

3) $\lim_{n \to \infty} h_{mn} = v_1$ *strongly in $H^k(\Omega_{1m})$, $k = 1, 2, \ldots$.*

Proof. 1) is proved in the same way as lemma 5. Observe that now $\| h^+_{mn} \|_{L^2(\Omega)} \leqq 1$ so we can only conclude that $0 \leqq c \leqq 1$. 2) From the definition of h_{mn} one sees that $h_{mn} 1_{\Omega \setminus \Omega_{1m}} = h^+_{mn} 1_{\Omega \setminus \Omega_{1m}}$ and that $\lim_{n \to \infty} h_{mn} 1_{\Omega \setminus \Omega_{1m}} = c \bar{v}_1 1_{\Omega \setminus \Omega_{1m}}$ strongly in $L^2(\Omega)$. On the other hand $(h_{mn} 1_{\Omega_{1m}})_{n \geqq m}$ is bounded in $H^2(\Omega_{1m})$. Thus there is a subsequence of $(h_{mn} 1_{\Omega_{1m}})_{n \geqq m}$ (denoted the same) such that $\lim_{n \to \infty} h_{mn} 1_{\Omega_{1m}}$ exists strongly in $L^2(\Omega_{1m})$. This proves that there exists $h \in L^2(\Omega)$, $\|h\|_{L^2(\Omega)} = 1$, with $\lim_{n \to \infty} h_{mn} = h$ strongly in $L^2(\Omega)$. In consequence $\lim_{n \to \infty} h^+_{mn} = h^+ = c \bar{v}_1$ and $\lim_{n \to \infty} h^-_{mn} = h^-$ strongly in $L^2(\Omega)$. Note that $\operatorname{supp} \bar{v}_1 = \bar{\Omega}_1$ and $\operatorname{supp} h^- \subset \bar{\Omega}_{1m} \subset \bar{\Omega}_1$. Thus only $c = 1$ or $c = 0$ are possible. But $c = 0$ is impossible because in this case $\| h^- \|_{L^2(\Omega)} = 1$, and this would contradict Corollary 6. So 2) and thus 3) follow.

To complete the proof of the theorem one now applies lemma 3 and 4 to the sequence $(h_{mn})_{n \in \mathbb{N}}$ to conclude that for every $\varepsilon > 0$ there exists a number n_0 such that for $n > n_0$ $h_{mn} \neq 0$ for all $x \in \Omega \setminus [\Gamma]_\varepsilon$ and hence also $u_{n2}(x) \neq 0$ for all $x \in \Omega \setminus [\Gamma]_\varepsilon$.

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3. Extensions and generalizations

I. Here we show that theorem 1 can be generalized to the zero set of the first N eigenfunctions of (1) in the following manner. Let $Z_j(q)=\{x \in \Omega;\, u_j(x)=0\}$, where $u_j \in H_0^1(\Omega) \cap H^2(\Omega)$ solves $-\Delta u_j + q u_j = \lambda_j u_j$, $2 \leqq j \leqq N$. Assume that there are Lipschitz continuous curves $\gamma_i\colon [0,1] \to \bar{\Omega}$, $1 \leqq i \leqq k$, with $k \leqq N$, such that $\Omega \backslash \Gamma$ has exactly N components, where $\Gamma = \bigcup_{i=1}^{k} \gamma_i$. Assume that Ω satisfies an interior sphere condition for all $x \in \partial\Omega$.

Theorem 2. *There exists for every $\varepsilon > 0$ a potential $q_\varepsilon \in L^\infty(\Omega)$ such that*

$$Z_j(q_\varepsilon) \subset [\Gamma]_\varepsilon, \quad 2 \leqq j \leqq N.$$

Proof. The proof follows the same lines as the proof of theorem 1, and we just point out the main differences.

Let $\Omega_1, \ldots, \Omega_N$ denote the components of $\Omega \backslash \Gamma$, and assume without loss of generality that $\mu_1(\Omega_1) < \mu_1(\Omega_2) < \cdots < \mu_1(\Omega_N)$, where $\mu_1(D)$ denote the first eigenvalue of

$$\text{(8)} \qquad \begin{aligned} -\Delta u &= \lambda u, \quad \text{in} \quad D, \\ u &= 0, \quad \text{on} \quad \partial D. \end{aligned}$$

By $v_i := v_1(\Omega_i)$, $i = 1, \ldots, N$, we denote the corresponding first eigenfunctions, with $\|v_i\|_{L^2(\Omega_i)} = 1$, and $v_i > 0$ in Ω_i. We set $[\Gamma]_{1/n} = \{x \in \Omega;\, \operatorname{dist}(\Gamma, x) < 1/n\}$ and let

$$\Omega_{ni} = \Omega_i \backslash [\Gamma]_{1/n} \quad i = 1, \ldots, N.$$

Set $\mu_{ni} := \mu_1(\Omega_{ni})$, $i = 1, \ldots, N$, and v_{ni} the corresponding first eigenfunctions (chosen positive and with $\|v_{ni}\|_{L^2(\Omega_{ni})} = 1$). The sequences $(\mu_{ni})_{n \geqq 1}$ are monotonically decreasing and $\mu_{ni} \to \mu_1(\Omega_i)$ for $n \to \infty$ $(1 \leqq i \leqq N)$. Now choose positive convergent sequences $(a_{ni})_{n \in \mathbb{N}}$ $(i = 1, \ldots, N-1)$ with $a_i = \lim_{n \to \infty} a_{ni}$, such that for

$$\eta_{ni} := \mu_{ni} + a_{ni} \text{ and } \eta_1(\Omega_i) = \mu_1(\Omega_i) + a_i$$

the following inequalities hold (for all $n \in \mathbb{N}$):

$$\eta_{n1} < \eta_{n2} < \cdots < \eta_{n(N-1)} < \mu_1(\Omega_N)$$

and

$$\eta_1(\Omega_i) = \mu_1(\Omega_N) < \eta_2(\Omega_i), \quad i = 1, \ldots, N-1.$$

Finally, let $q_n(x) = r_n 1_{[\Gamma]_{1/n}}(x) + \sum_{i=1}^{N-1} a_{ni} 1_{\Omega_{ni}}(x)$, where r_n are constants which will be chosen later. We denote by $\lambda_{n1} < \lambda_{n2} \leqq \lambda_{n3} \leqq \cdots$ the Dirichlet eigenvalues of

$$\text{(9)} \qquad -\Delta y + q_n y = \lambda y, \quad \text{in } \Omega, \quad y = 0 \text{ on } \partial\Omega,$$

and by $u_{n1}, u_{n2}, \ldots$, the corresponding $L^2(\Omega)$-normalized eigenfunctions.

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One now proves as in lemma 1 that for all $i \in \{1, \ldots, N\}$ and all $n \in \mathbb{N}$

$$0 \leqq \lambda_{ni} \leqq \mu_1(\Omega_N), \tag{10}$$

$$\|\nabla u_{ni}\|^2_{L^2(\Omega)} \leqq \mu_1(\Omega_N), \tag{11}$$

$$\|\Delta u_{n2}\|_{L^2(\Omega_{ni})} \leqq \sup\{(\alpha_{in}, \mu_1(\Omega_N)\}. \tag{12}$$

In fact, one uses again the variational characterization of the i-th eigenvalue $(1 \leqq i \leqq N)$ to obtain

$$\lambda_{ni} \leqq \sum_{j=1}^{i} \delta_{nj}^2 \left(\|\nabla v_{nj}\|^2_{L^2(\Omega_{nj})} + a_{nj}\|v_{nj}\|^2_{L^2(\Omega_{nj})}\right)$$

where $a_{nN} := 0$ $(n \in \mathbb{N})$ and $\delta_{nj} \in [-1, 1]$ are chosen such that

$$\sum_{j=1}^{i} \delta_{nj}^2 = 1 \text{ and } \sum_{j=1}^{i} \langle \delta_{nj} v_{nj}, u_{nk}\rangle = 0, \quad k = 1, \ldots, i-1.$$

One then gets

$$\lambda_{ni} \leqq \sum_{j=1}^{i} \delta_{nj}^2 \eta_{nj} \leqq \sum_{j=1}^{i} \delta_{nj}^2 \mu_1(\Omega_N) = \mu_1(\Omega_N).$$

From this (10) and (11) follow, while (12) follows from equation (9) restricted to the sets Ω_{ni}.

As in lemma 2 we denote by d_n and c_n the interpolation respectively embedding constants for $H^{1/2}([\Gamma]_{1/n})$. Choosing $(r_n)_{n \in \mathbb{N}}$ such that $\lim_{n \to \infty} \frac{c_n \cdot d_n}{r_n^{1/4}} = 0$ one now proves exactly as in lemma 2 that there exist for all $i \in \{1, \ldots, N\}$ constants $\alpha_j := \alpha_j(i)$, $j = 1, \ldots, N$, with $\sum_{j=1}^{N} \alpha_j^2 = 1$ such that

$$\lambda_i := \lim_{n \to \infty} \lambda_{ni} = \mu_1(\Omega_N), \tag{13}$$

$$u_i := \lim_{n \to \infty} u_{ni} = \sum_{j=1}^{N} \alpha_j v_j \quad \text{weakly in} \quad H_0^1(\Omega), \tag{14}$$

$$\lim_{n \to \infty} u_{ni} = \alpha_j v_j \quad \text{strongly in} \quad H^k(\Omega_{mj}), \quad (\text{all } m \text{ and } k). \tag{15}$$

Consider now u_i for some fixed $i \in \{2, \ldots, N\}$. Then $\alpha_j(i) \neq 0$ for at least one $j \in \{1, \ldots, N\}$. We can assume that $\alpha_j > 0$. One now shows as in lemma 3 that for an arbitrary open set U with $\bar{U} \subset \Omega_j$ there exists a number n_0 such that $\bar{U} \cap Z_{nj} = \emptyset$, for all $n \geqq n_0$, where $Z_{nj} = \{x \in \Omega;\, u_{nj}(x) = 0\}$. If $\alpha_k = 0$ for all $k \in \{1, \ldots, N\}$ then one easily completes the proof as in lemma 4.

Assume now that $\alpha_k = 0$ for some $k \in \{1, \ldots, N\}$. Let $\Omega_{mk} = \Omega_k \setminus [\Gamma]_{1/m}$ and let $F_n^+ = \{x \in \Omega;\, u_{ni}(x) > 0\}$ and $F_n^- = \{x \in \Omega;\, u_{ni}(x) < 0\}$. The sets F_n^+ and F_n^- consist possibly of several components which we denote by $F_{n,r}^+$ and $F_{n,s}^+$. Let now G_n denote

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that component of F_n^+ or F_n^- which intersects Ω_{mk}, and which satisfies

$$|G_n \cap \Omega_{mk}| \geqq \max_{r,s} \{|F_{n,r}^+ \cap \Omega_{mk}|, |F_{n,s}^- \cap \Omega_{m,k}|\},$$

where $|D|$ denotes the Lebesgue-measure of the set D. By choosing an appropriate subsequence, if necessary, we may assume that either $G_n \subset F_n^+$ or $G_n \subset F_n^-$ for all $n \in \mathbb{N}$. Assume that the first is the case.

Let now $\tilde{u}_{ni} := u_{ni} 1_{G_n}$ and $f_n := \tilde{u}_{ni} / \|\tilde{u}_{ni}\|_{L^2(\Omega)}$. As in lemma 5 one now proves that $\lim_{n\to\infty} f_n = \bar{v}_k$ weakly in $H_0^1(\Omega)$ and a.e. in Ω, where $\bar{v}_k$ denotes the extension by zero to the whole of Ω of v_k (the first Dirichlet-eigenfunction on Ω_k).

Also the statement corresponding to lemma 6 follows now easily, namely that $\Omega_k \setminus \bigcup_{n\in N} B_n$ is a null set, where $B_n := \bigcap_{j \geqq n} G_j$.

Finally, one introduces the functions

$$g_{mn} := u_{ni} 1_{\Omega_{mk} \cup G_n}, \quad h_{mn} := g_{mn} / g_{mn} \|_{L^2(\Omega)}$$

and proves the statements of lemma 7, which allows to conclude the proof.

Corollary. *For any given $N \in \mathbb{N}$ there exist potentials $q \in L^\infty(\Omega)$ such that the length of the nodal line of the first N eigenfunctions of* (1) *is arbitrarily long.*

Proof. Let $L > 0$ be given, and let γ_i, $i = 1, \ldots, N-1$, denote disjoint curves of length $l(\gamma_i) \geqq L+1$ and such that $\Omega \setminus \Gamma$ $\left(\text{with } \Gamma = \bigcup_{i=1}^{N-1} \gamma_i\right)$ has exactly N components. By theorem 2 we can find potentials q such that the nodal sets Z_i, $i = 2, \ldots, N$, lie in arbitrary ε-neighbourhoods of Γ. This clearly implies that we can find potentials q such that $l(Z_i) > L$, for $i = 2, \ldots, N$, where $l(Z_i) :=$ length of Z_i.

II. Suppose that

$$A = \sum_{i,j=1}^{2} \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial}{\partial x_j}$$

is a uniformly elliptic, formally selfadjoint linear differential expression of second order with real-valued coefficient functions $a_{ij} = a_{ji} \in C^1(\bar{\Omega})$, and let L denote the linear operator induced by A in $L^2(\Omega)$ with domain $D(L) = H_0^1(\Omega) \cap H^2(\Omega)$. Then L is selfadjoint and has a compact resolvent; the spectrum of L consists of a sequence $\lambda_1 < \lambda_2 \leqq \lambda_3 \leqq \cdots$ of eigenvalues with finite multiplicities.

It is a straightforward application of the general theory of elliptic PDE to prove theorem 1 and 2 for the equation

$$Au + qu = \lambda u, \quad \text{in} \quad \Omega$$
$$u = 0, \quad \text{on} \quad \partial\Omega.$$

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